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Jürgen Ehlgen

University of Siegen

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A Comparison of Solution Methods for Real Business Cycle Models*

Jürgen Ehlgen
University of Siegen
Department of Economics
D-57068 Siegen, Germany

Phone: (49) 271 740 3217
Fax: (49) 271 740 2310
E-mail: ehlgen@wap-server.fb5.uni-siegen.de

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Abstract

This paper discusses solution procedures for real business cycle (RBC) models. First, we show that the most often used solution methods, the linear-quadratic approximation, the Lagrange multiplier, and the Euler equation approach all lead to the same decision function. Second, we demonstrate that deterministic and stochastic detrending methods which are used to transform the growing model economy to a stationary one, lead to the same model solution, no matter if the technology process has a unit root or not. Third, we show that contrary to statements in the literature the numerical value of the growth rate of the model can have substantial effects on the model results.

Key words: Dynamic optimization; Euler equation; Real business cycles; Unit roots
JEL classification: C61, C63

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1. Introduction

RBC models can be solved exactly only under very special assumptions. The most prominent case is the model of Long and Plosser (1983) who suppose a logarithmic utility function and a Cobb-Douglas production function with 100% depreciation. If under more realistic assumptions the exact solution can not be found, then the model must be solved approximately. Numerous such solution methods are proposed in the literature.[1] Some of these methods are highly numerical and require the implementation of a contraction mapping operator on a computer. Examples are the value function iteration employed by Greenwood, Hercowitz, and Huffman (1988), Danthine, Donaldson, and Mehra (1989) and Christiano (1990a), and the Euler equation approach of Coleman (1990), den Haan and Marcet (1990), Tauchen (1990), and Judd (1992). On the other hand, there are solution methods which are less numerical and lead to linear approximative decision rules. A first approach in this class is the procedure introduced by Kydland and Prescott (1982) and extended to nonstationary economies by Christiano (1988), to eliminate the nonlinear constraints by substituting them into the utility function and then to take the second order Taylor approximation around the deterministic equilibrium point of the system. The resulting optimal linear regulator problem is usually solved by the iteration of the matrix Riccati equation.[2] The variant of McGrattan (1990) eliminates the nonlinearities by approximating the constraints by linear functions. A second approach proposed by King, Plosser, and Rebelo (1988a) is to linearize the first order conditions of the Lagrangian of the optimization problem. Chow (1992, 1993, 1997) in a series of contributions describes a similar procedure but proposes to linearize the constraints in advance.[3] In a third approach Campbell (1994) takes the first order Taylor approximation of the Euler equation of the model.

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1 See Taylor and Uhlig (1990) and Danthine and Donaldson (1995) for a survey of different solution methods.

2 This solution method is described in detail by Hansen and Sargent (1988).

3 Kwan and Chow (1997) use a numerical procedure to solve the first order conditions of the original problem. They describe the linearization of the constraints as a particular implementation of the Lagrange multiplier approach, which in general they prefer to value function iteration.

King, Plosser, and Rebelo (1988a, p. 211) state that their procedure is essentially equivalent to the procedure of Kydland and Prescott (1982), and Christiano (1990b, p. 99) calls the solution procedure of King, Plosser, and Rebelo (1988a) a linear-quadratic approximation. Campbell (1994) notes that in the case of homoskedastic shocks his solution method yields the same results as the linear-quadratic approximation approach of Christiano (1988). Reiter (1997) on the other hand shows for the simple deterministic Ramsey optimal growth model that the various solution procedures give different decision rules, depending on the choice of the state and the control variables. Therefore, the first objective of this paper is to compare these solution methods which all lead to linear decision rules, and to show under which conditions they give the same solution.[4]

While the first part of the paper deals with stationary models, it is the concern of the second part how to deal with the trend in RBC models. The solution procedures of King, Plosser, and Rebelo (1988a and 1988b) and Christiano (1988) requires to transform the growing model economy into a stationary one. Hansen (1997) in his comparative study uses different procedures for this transformation, depending on the nature of the technology process. If (the logarithm of) technology follows a trend-stationary stochastic process the growing variables are divided by the deterministic time trend (deterministic detrending), if technology follows a difference stationary process, the growing model variables are divided by the lagged technology variable (stochastic detrending). It is shown in the present paper that it is not necessary to apply different detrending methods because the deterministic and the stochastic detrending procedures lead to the same solution of the model.

Christiano (1988) and Hansen (1997) assume a logarithmic utility function, presumably because in the case, that the constant elasticity of intertemporal substitution is different from unity, a random walk term appears in the quadratic approximation even if the stochastic detrending method is used. On the other hand Campbell (1994) linearizes the Euler equation of the original nonstationary model with a general isoelastic utility function and a possibly stochastic trend. This procedure leads to a

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[4] Clearly, the solutions of the pure numerical methods depend strongly on the specific implementation of the procedure. Therefore a comparison with those solution methods is beyond the scope of this paper.

stationary stochastic difference equation, so that a restriction to the logarithmic case is not necessary. It is shown in the present paper that the deterministic and the stochastic detrending method both lead to the same Euler equation as the approach of Campbell (1994). Therefore, the most preferred methods to solve RBC models, first the Lagrange multiplier method of King, Plosser and Rebelo (1988a) and second the linear-quadratic approximation approach of Kydland and Prescott (1982), can be applied even for the case of a stochastic trend and a general isoelastic utility function. It is sufficient in this case to adjust the variables for the deterministic trend.

The paper also examines the effects of a variation in the growth rate of technology on the model solution. Hansen (1997) claims that a change in the growth rate does not affect the model statistics. But it is shown here that this result depends on a specific parameter variation of Hansen's procedure. In general the effects of a change in the growth rate depend upon the fact which of the remaining parameters also change. It is possible to change these parameters in such a way that the variation in the growth rate is nearly neutralized. But it is also possible to change the remaining parameters in such a manner that the effects on the model statistics are substantial.

The plan of the paper is as follows. Section 2 analyzes the three approaches to approximately solve stationary stochastic dynamic optimization problems and examines the significance of linear constraints. Section 3 compares the different methods to obtain stationary Euler equations from a nonstationary economy for the case of a deterministic and a stochastic trend. Section 4 investigates the effects of a variation in the growth rate on the model results. Section 5 gives a short summary of the paper.

2. Solution methods for stationary models

Let X_t be an $(n \times 1)$ vector of state variables in period t, U_t an $(m \times 1)$ vector of control variables, and Z_t an exogenous stationary stochastic vector process of dimension $(l \times 1)$. Suppose that the transition equation of the system, which relates the future state of the system to the current state, the control variable, and to the stochastic shock, can be solved for the control variable:[5]

$$U_t = g(X_t, X_{t+1}, Z_t). \tag{1}$$

The objective is to maximize

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t f(X_t, U_t)\right\} \tag{2}$$

subject to the transition equation (1) and given X_0, where $f(X_t, U_t)$ is the one-period return function, β is the discount factor, which is positive but less than unity, and E_t denotes the expectation operator conditioned on information available in period t. In the following subsections several procedures to solve the dynamic stochastic optimization problem are compared.

2.1. The Euler equation approach

Define the function r by

$$r(X,Y,Z) = f[X, g(X,Y,Z)]. \tag{3}$$

Then, substituting (1) into (2), we have to maximize

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t r(X_t, X_{t+1}, Z_t)\right\} \tag{4}$$

given X_0. The first order conditions of this optimization problem are the stochastic Euler equations[6]

$$r_Y(X_t, X_{t+1}, Z_t) + \beta E_t r_X(X_{t+1}, X_{t+2}, Z_{t+1}) = 0,\ t = 0, 1, 2, \ldots, \tag{5}$$

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[5] In this formulation of the stochastic optimization problem the decision is made after the realization of the stochastic shock, so that the state variable in the next period is known with certainty.

[6] See Stokey and Lucas (1989). It is assumed that the transversality condition and the second order conditions are also fulfilled.

where r_X and r_Y are the derivatives of $r(X,Y,Z)$ with respect to X and Y, respectively.[7] First consider the corresponding deterministic optimization problem with $Z_t \equiv Z^*$, where Z^* is the unconditional expectation of Z_t. Suppose the deterministic system has an unique equilibrium $X_t = X^*$, $t = 0, 1, \ldots$. From equation (5), the equilibrium point must satisfy

$$r_Y^* + \beta r_X^* = 0, \tag{6}$$

where $r_X^* = r_X(X^*, X^*, Z^*)$ and $r_Y^* = r_Y(X^*, X^*, Z^*)$. This is a system of n equations to solve for the n unknowns in X^*. The first order Taylor approximation of equation (5) around (X^*, X^*, Z^*) is

$$\beta r_{XY'}^* E_t x_{t+2} + (r_{YY'}^* + \beta r_{XX'}^*) x_{t+1} + r_{YX'}^* x_t = -(r_{XZ'}^* E_t z_{t+1} + r_{YZ'}^* z_t) \;. \tag{7}$$

Here, $x_t = X_t - X^*$ and $z_t = Z_t - Z^*$, and $r_{XX'}^*$, $r_{YY'}^*$, $r_{YX'}^* = (r_{XY'}^*)'$, $r_{XZ'}^*$, and $r_{YZ'}^*$ are the matrices of second order derivatives of $r(X,Y,Z)$, again evaluated at (X^*, X^*, Z^*).

Now let $U^* = g^* = g(X^*, X^*, Z^*)$ and denote the derivatives of $f(X,U)$ at (X^*, U^*) and the derivatives of $g(X,Y,Z)$ at (X^*, X^*, Z^*) similar to the derivatives of $r(X,Y,Z)$. Applying the chain rule and the product rule gives the following relationships between the derivatives of r and the derivatives of f and g:

$$r_X^* = f_X^* + g_X'^* f_U^*, \tag{8a}$$

$$r_Y^* = g_Y'^* f_U^*, \tag{8b}$$

and

$$r_{XX'}^* = f_{XX'}^* + f_{XU'}^* g_{X'}^* + g_{XX'}'^* (I \otimes f_U^*) + g_X'^* (f_{UX'}^* + f_{UU'}^* g_{X'}^*), \tag{9a}$$

$$r_{XY'}^* = f_{XU'}^* g_{Y'}^* + g_{XY'}'^* (I \otimes f_U^*) + g_X'^* f_{UU'}^* g_{Y'}^*, \tag{9b}$$

$$r_{XZ'}^* = f_{XU'}^* g_{Z'}^* + g_{XZ'}'^* (I \otimes f_U^*) + g_X'^* f_{UU'}^* g_{Z'}^*, \tag{9c}$$

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[7] See appendix 1 for the collection of some differentiation rules for vector functions which are employed in this paper.

$$r^{*}_{YY'} = g'^{*}_{YY'}\left(I \otimes f^{*}_{U}\right) + g'^{*}_{Y} f^{*}_{UU'} g^{*}_{Y'}, \tag{9d}$$

$$r^{*}_{YZ'} = g'^{*}_{YZ'}\left(I \otimes f^{*}_{U}\right) + g'^{*}_{Y} f^{*}_{UU'} g^{*}_{Z'}. \tag{9e}$$

2.2. The Lagrange multiplier approach

The Lagrangian of the optimization problem (1) and (2) is given by

$$L = E_0 \sum_{t=0}^{\infty} \left\{ \beta^t f(X_t, U_t) - \tilde{\Lambda}'_t \left[U_t - g(X_t, X_{t+1}, Z_t) \right] \right\}, \tag{10}$$

where $\tilde{\Lambda}_t$ is an $(m \times 1)$ vector of Lagrange multipliers. With $\Lambda_t = \tilde{\Lambda}_t / \beta^t$ the first order conditions of equation (10) are[8]

$$\Lambda_t = f_U(X_t, U_t), \tag{11}$$

$$g'_Y(X_t, X_{t+1}, Z_t)\Lambda_t + \beta E_t\left[f_X(X_{t+1}, U_{t+1}) + g'_X(X_{t+1}, X_{t+2}, Z_{t+1})\Lambda_{t+1} \right] = 0, \tag{12}$$

and the transition equation (1). The deterministic equilibrium values must satisfy

$$g'^{*}_{Y} f^{*}_{U} + \beta\left[f^{*}_{X} + g'^{*}_{X} f^{*}_{U} \right] = 0. \tag{13}$$

It is seen from equations (8) that (13) is identical to (6), so that the same deterministic equilibrium is obtained. With $\Lambda^* = f^*_U$ and $\lambda_t = \Lambda_t - \Lambda^*$ the linearized first order conditions are

$$\lambda_t = f^{*}_{UX'} x_t + f^{*}_{UU'} u_t, \tag{14}$$

$$\begin{aligned} & g'^{*}_{YX'}(I \otimes \Lambda^*)x_t + g'^{*}_{YY'}(I \otimes \Lambda^*)x_{t+1} + g'^{*}_{YZ'}(I \otimes \Lambda^*)z_t + g'^{*}_{Y}\lambda_t + \\ & \beta E_t\left\{ \left[f^{*}_{XX'} x_{t+1} + f^{*}_{XU'} u_{t+1} \right] + \left[g'^{*}_{XX'}(I \otimes \Lambda^*)x_{t+1} + g'^{*}_{XY'}(I \otimes \Lambda^*)x_{t+2} + g'^{*}_{XZ'}(I \otimes \Lambda^*)z_{t+1} + g'^{*}_{X}\lambda_{t+1} \right] \right\} = 0, \end{aligned} \tag{15}$$

$$u_t = g^{*}_{X'} x_t + g^{*}_{Y'} x_{t+1} + g^{*}_{Z'} z_t. \tag{16}$$

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[8] See Chow (1997).

Eliminating u_t, λ_t, and Λ^* in equations (14) to (16), rearranging terms, and using equations (9), again gives equation (7). So the Lagrange multiplier approach leads to the same solution of the optimization problem as the Euler equation approach.

2.3. Linearizing the transition equation

Now maximize the objective function (2) subject to the linearized transition equation

$$U_t = \tilde{g}(X_t, X_{t+1}, Z_t) = g^* + g^*_{X'}(X_t - X^*) + g^*_{Y'}(X_{t+1} - X^*) + g^*_{Z'}(Z_t - Z^*). \tag{17}$$

Proceeding as before, define $\tilde{r}(X,Y,Z) = f[X, \tilde{g}(X,Y,Z)]$. Then the linearized Euler equation is given by

$$\beta \tilde{r}^*_{XY'} E_t x_{t+2} + (\tilde{r}^*_{YY'} + \beta \tilde{r}^*_{XX'}) x_{t+1} + \tilde{r}^*_{YX'} x_t = -(\beta \tilde{r}^*_{XZ'} E_t z_{t+1} + \tilde{r}^*_{YZ'} z_t). \tag{18}$$

Note that at the deterministic equilibrium the first order derivatives of $\tilde{g}$ are equal to those of g, and the second order derivatives of $\tilde{g}$ are equal to zero. So the second order derivatives of $\tilde{r}$ at the equilibrium point are identical to the second order derivatives of r, except that the second order derivatives of g do not appear. Therefore, one obtains equation (18) if in equations (7) and (9) the second order derivatives of g are set to zero. Equation (18) is not identical to equation (7) because in general it makes a difference if an equation is first linearized and then differentiated or, alternatively, first differentiated and then linearized.

2.4. The linear-quadratic approximation

Now maximize the quadratic approximation of the return function

$$\tilde{f}(X_t, U_t) = f^* + f^*_{X'} x_t + f^*_{U'} u_t + \frac{1}{2} x_t' f^*_{XX'} x_t + \frac{1}{2} u_t' f^*_{UU'} u_t + x_t' f^*_{XU'} u_t \tag{19}$$

subject to the linearized transition equation (17). Because at the deterministic equilibrium the second derivatives of $\tilde{f}$ are identical to those of f, the Euler equation of the new optimization problem is again given by equation (18). But note that in the present case equation (18) is the exact Euler equation (of the approximated problem) because the first order conditions are already linear and no further approximation is required.

2.5. Conclusions

If the one-period return function is quadratic, and the transition equation and the law of motion of the exogenous stochastic process are linear, then the Lagrange method and the dynamic programming method lead to the same optimal policy function.[9] It follows that, if the transition equation and the equation for the stochastic process are linear, then the three solution methods: (1) linearizing the Euler equation, (2) linearizing the first order conditions of the Lagrangian, and (3) dynamic programming with a quadratic approximation of the return function all lead to the same solution. If the solution procedure requires a linear transition equation, as the optimal linear regulator problem does, then all the nonlinear constraints should first be substituted into the return function to eliminate the nonlinearities. This is the proceeding of, for example, Kydland and Prescott (1982) and Christiano (1988) and emphasized by Hansen and Prescott (1994). If the transition equation is nonlinear but approximated by a linear function, as proposed by McGrattan (1990) and Chow (1997), then there is a loss in accuracy because the second derivatives of the transition equation are neglected. So the conclusion of Reiter (1997) for the simple deterministic Ramsey optimal growth model, that the problem should be formulated in such a way that the constraints are linear, holds in general.

3. Stochastic and deterministic trends in a RBC model

Consider the following RBC model. The representative infinitely-lived agent maximizes

$$E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\theta} \right)^t u(C_t, L_t), \tag{20}$$

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[9] See Chow (1975) or Hansen and Sargent (1997).

where again E_t is the expectation operator conditioned on information available at time t, C_t is consumption and L_t is leisure in period t, and θ is the positive rate of time preference. The period utility function is given by

$$u(C,L)=\begin{cases}\dfrac{1}{1-\tau}C^{1-\tau}\exp\{(1-\tau)\omega(L)\} & \tau\neq 1\\ & \text{for}\\ \log(C)+\omega(L) & \tau=1\end{cases} \tag{21}$$

In (21) τ is the positive elasticity of the marginal utility of consumption, and $\omega(L)$ is a function with $\omega'(L)>0$ and $\omega''(L)\leq 0$. The period utility function satisfies the conditions given in King, Plosser, and Rebelo (1988a) and Barro and Sala-i-Martin (1995), that must be fulfilled for the existence of a deterministic balanced growth path. The single good is produced according to a linearly homogenous production function with labor-augmenting technical progress:

$$Y_t=F(K_t,A_tN_t). \tag{22}$$

Here, Y_t is output, K_t is the predetermined stock of capital, N_t is labor input, and A_t is a stochastic process, which describes the level of the technology. Harrod-neutral technical progress is required for the existence of a deterministic balanced growth path. The law of motion for the capital stock is given by

$$K_{t+1}=(1-\delta)K_t+I_t, \tag{23}$$

where I_t is investment and δ is the constant rate of depreciation, which lies between zero and unity. The household maximizes his expected lifetime utility subject to the resource constraints

$$L_t+N_t\leq 1 \tag{24}$$

and

$$C_t+I_t\leq Y_t. \tag{25}$$

It is assumed that the productivity level evolves according to

$$A_t=(1+\mu)^t\exp\{\tilde{A}_t\} \tag{26}$$

with $\mu \geq 0$. It is assumed that $\tilde{A}_t$ follows the autoregressive stochastic process

$$\phi(B)\tilde{A}_t = \varepsilon_t, \tag{27}$$

where $\phi(B)$ is a finite polynomial in the backshift operator B and ε_t is a zero mean white noise process with finite variance σ_ε^2. The roots of $\phi(B)$ determine the persistence of a shock. It is assumed that at most one root of $\phi(B)$ is equal to unity, the remaining roots are lying outside of the unit circle. If unity is a root, then $\tilde{A}_t$ follows a difference stationary stochastic process, if not, then $\tilde{A}_t$ follows a trend-stationary process. In both cases, the mean growth rate of the productivity level is given by μ.

If μ is positive the solution methods described in the foregoing section are not applicable directly because the variable A_t is growing over time. It follows that the model has no deterministic equilibrium. Instead King, Plosser, and Rebelo (1988a) show that on the balanced deterministic growth path with $\sigma_\varepsilon^2 = 0$, A_t, C_t, I_t, K_t, and Y_t are growing with the same rate μ. But due to the constant intertemporal elasticity of consumption and the homogeneity of the production function it is possible to adjust these variables for the trend, so that a deterministic equilibrium exists for the transformed variables. If we substitute the constraints (22) to (25) into the utility function (20) we can define the function R by

$$\begin{aligned} R(K_t, K_{t+1}, N_t, A_t) &= u(C_t, L_t) \\ &= u\left[F(K_t, A_t N_t) + (1-\delta)K_t - K_{t+1}; 1 - N_t\right]. \end{aligned} \tag{28}$$

Note that R is homogeneous of degree $1-\tau$ in its first, second, and fourth argument, i.e.

$$R(\nu K_t, \nu K_{t+1}, N_t, \nu A_t) = \nu^{1-\tau} R(K_t, K_{t+1}, N_t, A_t), \quad \nu > 0.\text{[10]} \tag{29}$$

As shown below this property of R is of crucial importance to obtain a solution of the model.

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[10] In the logarithmic case there appears an additive term $\log(\nu)$. But this term can be neglected in the following transformations because it does not affect the first order conditions.

3.1. The case of a deterministic trend

In the case of a deterministic trend we can stationarize the model by dividing the growing variables by the trend. If we define $\tilde{K}_t = \log\left[K_t/(1+\mu)^t\right]$, $\tilde{N}_t = \log(N_t)$, and $r(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t) = R\left[\exp\{\tilde{K}_t\}, (1+\mu)\exp\{\tilde{K}_{t+1}\}, \exp\{\tilde{N}_t\}, \exp\{\tilde{A}_t\}\right]$, we can write[11]

$$\left(\frac{1}{1+\theta}\right)^t R(K_t, K_{t+1}, N_t, A_t) = \beta^t r(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t)\,, \tag{30}$$

where $\beta = (1+\mu)^{1-\tau}/(1+\theta)$. If we define the state vector of the model as $\left[\tilde{K}_t \quad \tilde{N}_{t-1}\right]'$ and the control vector as $\left[\tilde{K}_{t+1} \quad \tilde{N}_t\right]'$, we have the model of the foregoing section with the period return function (30) and the linear stochastic process (27). The transition equation is also linear and says that the next period state variable is equal to the control variable. The Euler equations of the RBC model are given by

$$r_2(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t) + \beta E_t\left[r_1(\tilde{K}_{t+1}, \tilde{K}_{t+2}, \tilde{N}_{t+1}, \tilde{A}_{t+1})\right] = 0, \tag{31a}$$

$$r_3(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t) = 0, \tag{31b}$$

where r_i is the derivative of r with respect to its i-th argument. The deterministic equilibrium value of $\tilde{A}_t$ is equal to zero. With $\tilde{K}^*$ and $\tilde{N}^*$ the deterministic equilibrium values of $\tilde{K}_t$ and $\tilde{N}_t$, and $r_i^* = r_i(\tilde{K}^*, \tilde{K}^*, \tilde{N}^*, 0)$, we have the two equations $r_2^* + \beta r_1^* = 0$ and $r_3^* = 0$ to solve for $\tilde{K}^*$ and $\tilde{N}^*$.[12] Then the linearized Euler equations are

$$\beta r_{12}^* E_t k_{t+2} + (r_{22}^* + \beta r_{11}^*) k_{t+1} + r_{12}^* k_t + \beta r_{13}^* E_t n_{t+1} + r_{23}^* n_t + \beta r_{14}^* E_t a_{t+1} + r_{24}^* a_t = 0, \tag{32a}$$

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[11] We also take the logarithm of the variables, because it is shown by Christiano (1988) and Taylor and Uhlig (1990) that this transformation gives a better approximation.

[12] It is assumed that the two equations have a unique solution. In some cases the equations may be highly nonlinear so that the deterministic equilibrium values must be computed with numerical methods. Alternatively the equilibrium may be given, from which parameters of the utility function or the production function are determined.

$$r_{23}^{*}k_{t+1}+r_{13}^{*}k_{t}+r_{33}^{*}n_{t}+r_{34}^{*}a_{t}=0, \tag{32b}$$

with $k_t = \tilde{K}_t - \tilde{K}^*$, $n_t = \tilde{N}_t - \tilde{N}^*$, and $a_t = \tilde{A}_t$.

3.2. The case of a stochastic trend

Christiano (1988), King, Plosser, and Rebelo (1988b), and Hansen (1997) investigate the case that the stochastic process for $\tilde{A}_t$ has a unit root. Hansen (1997, p. 1010) argues, that in this case a different solution procedure should be employed, because if $\tilde{A}_t$ "follows a random walk, simulated variables may wander far from their steady-state values" and the "solution may be a poor one because the approximation may be accurate only when the variables take on values close to the steady state." Christiano (1988) and Hansen (1997) propose to transform the model by dividing the growing variables by the lagged technology variable A_t. Then it follows from equations (29) and (30) that

$$\left(\frac{1}{1+\theta}\right)^{t}R(K_t,K_{t+1},N_t,A_t)=\beta^{t}\exp\{(1-\tau)\tilde{A}_{t-1}\}r(\tilde{K}_t-\tilde{A}_{t-1},\tilde{K}_{t+1}-\tilde{A}_t+\Delta\tilde{A}_t,\tilde{N}_t,\Delta\tilde{A}_t). \tag{33}$$

Two things should be noted about this equation. First, in the logarithmic case with τ equal to one, $\tilde{A}_t$ disappears in equation (33). Therefore the transformed model is an optimization problem in the stationary variables $(\tilde{K}_t - \tilde{A}_{t-1})$, $\tilde{N}_t$, and $\Delta\tilde{A}_t$. Because the deterministic equilibrium value of $\Delta\tilde{A}_t$ is equal to zero, it follows from the Euler equations, that the deterministic steady state values of $(\tilde{K}_t - \tilde{A}_{t-1})$ and $\tilde{N}_t$ are equal to $\tilde{K}^*$ and $\tilde{N}^*$, respectively, so that in the deterministic and in the stochastic trend model the Taylor approximation is taken around the same values. Because $r(\tilde{K}_t,\tilde{K}_{t+1},\tilde{N}_t,\tilde{A}_t)$ is equal to $r(\tilde{K}_t-\tilde{A}_{t-1},\tilde{K}_{t+1}-\tilde{A}_t+\Delta\tilde{A}_t,\tilde{N}_t,0)$, the two functions of course have the same Taylor approximations. It follows that although the variables in (32) are nonstationary, the equations can be transformed in such a way that only stationary variables remain. The same statement holds for the resulting optimal decision functions which may be expressed in stationary variables. Then, these stationary variables will not wander far from their steady state values. Therefore in the logarithmic case there is no need for a special transformation if the technology process has a unit root. Even in this case it is sufficient to adjust the variables for the deterministic trend.[13]

Second, for $\tau \neq 1$ the nonstationary variable $\tilde{A}_t$ appears in equation (33) and the criticism of Hansen (1997) could be valid.[14] This seems to be the reason for Christiano (1988) and Hansen (1997) to investigate the logarithmic case only. Christiano (p. 252) explains this explicitly: "The solution strategy I use to solve the model requires that the function relating consumption to instantaneous utility have the property of converting multiplication into addition, as the logarithmic does."[15]

An alternative to explicitly stationarize the model is to take the Euler equations of the original nonstationary economy as proposed by Campbell (1994). This leads to the first order conditions

$$R_2(K_t, K_{t+1}, N_t, A_t) + \frac{1}{1+\theta} E_t R_1(K_{t+1}, K_{t+2}, N_{t+1}, A_{t+1}) = 0, \tag{34a}$$

$$R_3(K_t, K_{t+1}, N_t, A_t) = 0. \tag{34b}$$

Because R is homogenous of degree $1-\tau$ (in its first, second, and fourth argument), the derivatives R_1 and R_2 are homogenous of degree $-\tau$ and R_3 is homogenous of degree $1-\tau$. Therefore the Euler equations can be written as

$$A_t^{-\tau} R_2\left(\frac{K_t}{A_t}, \frac{K_{t+1}}{A_t}, N_t, 1\right) + \frac{1}{1+\theta} E_t\left\{A_{t+1}^{-\tau} R_1\left(\frac{K_{t+1}}{A_{t+1}}, \frac{K_{t+2}}{A_{t+1}}, N_{t+1}, 1\right)\right\} = 0, \tag{35a}$$

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[13] Indeed, Hansen (1997, p. 1013) reports that he obtains the same summary statistics for both procedures, but he attributes this result to the quality of the quadratic approximation.

[14] The same point holds for the procedure of King, Plosser, and Rebelo (1988b), who divide the model variables by the permanent component of technology.

[15] Remember that in the logarithmic case in the transformed return function an additive term appears. This term contains the exogenous stochastic process and can be neglected because it is not under control of the agent.

$$A_t^{-\tau} R_3\left(\frac{K_t}{A_t}, \frac{K_{t+1}}{A_t}, N_t, 1\right) = 0. \tag{35b}$$

Because

$$r_1\left(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t\right) = R_1\left[\exp\{\tilde{K}_t\}, (1+\mu)\exp\{\tilde{K}_{t+1}\}, \exp\{\tilde{N}_t\}, \exp\{\tilde{Z}_t\}\right]\exp\{\tilde{K}_t\},$$

$$r_2\left(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t\right) = R_2\left[\exp\{\tilde{K}_t\}, (1+\mu)\exp\{\tilde{K}_{t+1}\}, \exp\{\tilde{N}_t\}, \exp\{\tilde{Z}_t\}\right](1+\mu)\exp\{\tilde{K}_{t+1}\},$$

$$r_3\left(\tilde{K}_t, \tilde{K}_{t+1}, \tilde{N}_t, \tilde{A}_t\right) = R_3\left[\exp\{\tilde{K}_t\}, (1+\mu)\exp\{\tilde{K}_{t+1}\}, \exp\{\tilde{N}_t\}, \exp\{\tilde{Z}_t\}\right]\exp\{\tilde{N}_t\},$$

the equations (35) can be transformed to

$$r_2\left(\tilde{K}_t - \tilde{A}_t, \tilde{K}_{t+1} - \tilde{A}_t, \tilde{N}_t, 0\right) + \beta E_t\left[\exp\{(1-\tau)\Delta\tilde{A}_t\} r_1\left(\tilde{K}_{t+1} - \tilde{A}_{t+1}, \tilde{K}_{t+2} - \tilde{A}_{t+1}, \tilde{N}_{t+1}, 0\right)\right] = 0, \tag{36a}$$

$$r_3\left(\tilde{K}_t - \tilde{A}_t, \tilde{K}_{t+1} - \tilde{A}_t, \tilde{N}_t, 0\right) = 0. \tag{36b}$$

Because $\tilde{K}_t - \tilde{A}_t = \tilde{K}_t - \tilde{A}_{t-1} - \Delta\tilde{A}_t$, we have two equations in the stationary variables $\tilde{K}_t - \tilde{A}_{t-1}$, $\tilde{N}_t$, and $\Delta\tilde{A}_t$. That means that the Euler equations of the nonstationary RBC model are stationary stochastic difference equations. The deterministic steady state values of $\left(\tilde{K}_t - \tilde{A}_{t-1}\right)$ and $\tilde{N}_t$ again are equal to $\tilde{K}^*$ and $\tilde{N}^*$, respectively. Then the linearized Euler equations are

$$\begin{aligned}&\beta r_{12}^* E_t\left(k_{t+2} - a_{t+1}\right) + \left(r_{22}^* + \beta r_{11}^*\right)\left(k_{t+1} - a_t\right) + r_{12}^*\left(k_t - a_{t-1}\right) + \beta r_{13}^* E_t n_{t+1} + r_{23}^* n_t + \\ &\beta\left[(1-\tau) r_1^* - r_{11}^*\right] E_t \Delta a_{t+1} - r_{12}^* \Delta a_t = 0\end{aligned}, \tag{37a}$$

$$r_{23}^*\left(k_{t+1} - a_t\right) + r_{13}^*\left(k_t - a_{t-1}\right) + r_{33}^* n_t - r_{13}^* \Delta a_t = 0. \tag{37b}$$

Because only stationary variables appear in equations (37), the linear approximations should not be inaccurate. But the deterministic detrending procedure leads to the same linearized Euler equations, which can be shown as follows. Due to the homogeneity of R, we have the following version of Euler's theorem for homogenous functions: $(1-\tau) r_i^* = r_{i1}^* + r_{i2}^* + r_{i4}^*$, i = 1, 2, 3. Then it follows $r_{14}^* = (1-\tau) r_1^* - r_{11}^* - r_{12}^*$, $r_{24}^* = -\beta(1-\tau) r_1^* - r_{21}^* - r_{22}^*$, and $r_{34}^* = -r_{31}^* - r_{32}^*$ because $r_2^* + \beta r_1^* = 0$ and $r_3^* = 0$. Substituting this into equations (32) gives equations (37). Therefore, to stationarize the model it is sufficient to detrend the growing variables by the deterministic trend. It is not necessary to use a different procedure in the case of a stochastic trend even if the technology process has a unit root and the utility function is of the general isoelastic kind.

As an example take the RBC model in Hansen (1997) with a negative autocorrelation for the growth rate of the technology variable. In this model the subutility function for leisure is $\omega(L) = B \cdot L$ with $B > 0$, the production function is $F(K, AN) = K^{1-\alpha}(AN)^{\alpha}$ with $0 < \alpha < 1$, and the parameters are given by $\alpha = 0.64$, $\delta = 0.02$, $\rho = 0.01$, $\tau = 1$, $\mu = 0.005$, $\phi_1 = 1$, and $\phi_2 = -0.2$. Further, steady state labor is $N^* = \exp\{\tilde{N}^*\} = 0.3$, which determines $B = 2.868$. The adjustment for the deterministic trend, the transformation employed in section 3.1, gives the optimal decision functions[16]

$$\tilde{K}_{t+1} = 0.9420\tilde{K}_t + 0.0738\tilde{A}_t - 0.0158\tilde{A}_{t-1} + 0.1410,$$

$$\tilde{N}_t = -0.4770\tilde{K}_t + 0.6561\tilde{A}_t - 0.1792\tilde{A}_{t-1} - 0.0444.$$

Because the sum of coefficients in the first equation is equal to unity and in the second equation equal to zero,[17] the equations can be transformed to

$$(\tilde{K}_{t+1} - \tilde{A}_t) = 0.9420(\tilde{K}_t - \tilde{A}_{t-1}) - 0.9262(\tilde{A}_t - \tilde{A}_{t-1}) + 0.1410,$$

$$\tilde{N}_t = -0.4770(\tilde{K}_t - \tilde{A}_{t-1}) + 0.6561(\tilde{A}_t - \tilde{A}_{t-1}) - 0.0444.$$

These are nearly exactly the optimal decision functions obtained by Hansen (1997) with the alternative stochastic detrending transformation.[18]

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[16] The linear-quadratic approximation method was used for the solution of the model. The optimal linear regulator problem was solved with the procedure described in Ehlgen (1997a).

[17] Campbell (1994) obtains the same result for the unit root case.

[18] There may be several reasons for the slight differences in the coefficients. First, the utility in this paper depends on leisure, while the utility function in Hansen is formulated in labor. Second, there might be small differences in the numerical values of the parameters.

4. The importance of the trend for model statistics

In a parameter variation Hansen (1997) sets the growth rate of the model equal to zero and obtains the same summary statistics as in the model with a positive growth rate. He concludes that "abstracting from growth does not affect the properties of the fluctuations displayed by a RBC model with exogenous technical progress. This is reassuring given that so much of the business cycle literature abstracts from growth" (p. 1013). It is demonstrated in this section that this statement is only true in special circumstances.

Campbell (1994) shows that on the balanced growth path the relationship $(1+\mu)^{\tau}=(1+r)/(1+\rho)$ holds. Here r is the steady state interest rate determined by the marginal product of capital. Note that for $\tau=1$ we have approximately $r=\mu+\rho$. Further Campbell shows that the coefficients in the optimal decision functions depend on μ, δ, and r through the parameters $\lambda_1=(1+r)/(1+\mu)$, $\lambda_2=\alpha(r+\delta)/(1+\mu)$, and $\lambda_3=\alpha(r+\delta)/(1+r)$. Now, if μ is set to zero, δ increased by the change of μ, and ρ is not changed, then for $r=\mu+\rho$ to hold, r must decline with μ. It follows that λ_1, λ_2, λ_3, the optimal decision functions, and the model statistics change only slightly. Table 1, which gives the variabilities of the different variables of the RBC model in section 3.3. for several parameter constellations, illustrates this point. Case 1 is the base model of Hansen (1997) with a deterministic trend, and case 2 is the model with the growth rate set to zero. The variabilities are nearly identical.

But if the growth rate is modified and the changes in the remaining model parameters do not just offset the change in the growth rate in this special way, then the model statistics are not invariant to the growth rate. If, as in case 3, μ is set to zero but δ is not changed, then again r must go down. But now the coefficients λ_2 and λ_3 change as well as do the volatilities of the variables.

While these results do not seem to be of quantitative importance, the changes in the model statistics are more severe if the change in the growth rate affects the elasticity of intertemporal substitution but leaves the remaining parameters of the model unchanged.[19] Cases 4 and 5 are parameter variations for the deterministic trend model with a smaller growth rate and a higher elasticity of the marginal utility of consumption. Cases 6 to 8 examine the situation, that the logarithm of technology follows a stochastic trend. Case 6 is the model of Hansen (1997) with a logarithmic utility function and a first order autoregressive process with negative serial correlation for the growth rate of technology. Similarly to cases 4 and 5, in cases 7 and 8 the growth rate is reduced. Two main results follow. First, it turns out that the less the growth rate, the greater is the relative volatility of consumption and the less is the relative volatility of investment. Second, the changes in model statistics are much greater if the technology process has a unit root. If e.g. the growth rate is $\mu = 0.001$ (case 8) and accordingly $\tau = 5$, then the standard deviations of consumption and investment are not very different from the standard deviation of output.

Therefore, in general a variation in the growth rate can have important consequences for the model results. The strength of these effects depends on the nature of the stochastic process and on the fact which parameters bear the consequences of the variation in the growth rate. If the steady state interest rate and the rate of depreciation are allowed to vary and can "neutralize" the change in the growth rate, as in Hansen (1997), the effects may be negligible. But if the interest rate and the rate of depreciation are given, as in Campbell (1994), and the elasticity of intertemporal substitution changes, then the effects of a variation in the growth rate on the model results may be substantial. It follows that the treatment of the trend, especially the nature of the stochastic process and the numerical value of the growth rate, deserves a special attention.

5. Summary

In the present paper we first presented some solution methods for RBC models in a unified framework and verified the statements in e.g. King, Plosser, and Rebelo (1988a), Christiano (1990b),

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[19] If τ is different from unity the model can not be interpreted as an economy with indivisible labor (see Hansen and Prescott, 1995). The simple parameter variation here is made for illustrative reasons only.

and Campbell (1994) concerning the equivalence of the different solution methods. Second we showed that in a growing economy it is sufficient to adjust the variables for the deterministic trend even in the case that technology follows a stochastic trend. This result justifies the proceeding in e.g. Baxter and Crucini (1995) who use the deterministic detrending method in a model with a stochastic trend. It is not necessary to rely on the borderline case of a deterministic trend and an autoregressive root near unity as e.g. in Baxter and Crucini (1993). Third we demonstrated that a change in the growth rate of the model can influence the model results. Therefore not only the kind of the stochastic process, i.e. a trend- or a difference stationary process, but also the numerical value of the growth rate can have substantial effects upon the model statistics.

Table 1
Effects of Parameter Variations on Variabilities

	Parameter Values						Variabilities				
Case	ϕ_1	ϕ_2	μ	δ	τ	r	Y	C	I	N	K
1	0.95	0.0	0.0050	0.020	1	0.015	1.81	0.53	5.80	1.38	0.50
							1.00	0.29	3.20	0.76	0.28
2	0.95	0.0	0.0000	0.025	1	0.010	1.81	0.53	5.79	1.38	0.50
							1.00	0.29	3.20	0.76	0.28
3	0.95	0.0	0.0000	0.020	1	0.010	1.86	0.48	6.49	1.46	0.45
							1.00	0.26	3.49	0.78	0.24
4	0.95	0.0	0.0025	0.020	2	0.015	1.40	0.67	3.86	0.74	0.31
							1.00	0.48	2.75	0.53	0.22
5	0.95	0.0	0.001	0.02	5	0.015	1.30	0.72	3.44	0.59	0.26
							1.00	0.55	2.64	0.45	0.20
6	1.00	-0.2	0.0050	0.020	1	0.015	1.26	0.60	3.35	0.73	0.25
							1.00	0.48	2.65	0.58	0.20
7	1.00	-0.2	0.0025	0,020	2	0.015	0.96	0.74	1.76	0.25	0.12
							1.00	0.77	1.83	0.26	0.12
8	1.00	-0.2	0.0010	0.020	5	0.015	0.84	0.80	1.06	0.11	0.06
							1.00	0.96	1.27	0.13	0.07

The remaining parameter values in all cases are $\alpha=0.64$, $\rho=0.01$, and $B=2.868$. The first measure of variability is the standard deviation of the logarithm of the series after HP-filtering, multiplied by the factor 100. The loglinearized equation for output, investment, and consumption are derived in appendix 2. This leads to a state space system as in King, Plosser, and Rebelo (1988a). Then exact moments of the filtered series are computed with the procedure described in Ehlgen (1997b). The second measure of variability is the standard deviation relative to the standard deviation of Y.

Appendix 1

In this appendix we collect some rules for differentiation of vector functions. In the following, k, m, and n are positive integers and x is a real $(n \times 1)$ vector.

Let $F = (f_{ij})$ be an $(m \times k)$ matrix of single valued functions on the n-dimensional real space. Then the derivative $\partial F(x)/\partial x$ is defined as the $(mn \times k)$ matrix of functions $G = (g_{ij})$, such that $g_{m(s-1)+i,}(x) = \partial f_{ij}(x)/\partial x_s$ for $i = 1, \ldots, m$; $j = 1, \ldots k$; and $s = 1, \ldots, n$; where x_s is the s-th element of x. The derivative $\partial F(x)/\partial x'$ is defined as the $(m \times kn)$ matrix of functions $H = (h_{ij})$, such that $h_{i,k(s-1)+j}(x) = \partial f_{ij}(x)/\partial x_s$ for $i = 1, \ldots, m$; $j = 1, \ldots k$; and $s = 1, \ldots, n$. It is easy to show that $\partial F(x)/\partial x' = [\partial F'(x)/\partial x]'$. Note the special cases that the derivative of an $(m \times 1)$ column vector with respect to an $(1 \times n)$ row vector is an $(m \times n)$ matrix, and the derivative of an $(1 \times k)$ row vector with respect to an $(n \times 1)$ column vector is an $(n \times k)$ matrix.

With this convention it is easy to verify the following chain rule for vector functions. Let $f(u)$ be a vector of single valued functions on the m-dimensional real space and let $g(x)$ be an $(m \times 1)$ vector of single valued functions on the n-dimensional real space. Then:

$$\frac{\partial}{\partial x} f'[g(x)] = \frac{\partial}{\partial x} g'(x) \cdot \frac{\partial}{\partial u} f'(u), \text{ with } u = g(x).$$

Also we have the following version of the product rule. Let $G = (g_{ij})$ be an $(m \times k)$ matrix and let $r = (r_i)$ be an $(k \times 1)$ vector of single valued functions on the n-dimensional real space. Further, let $f = (f_i)$ be the $(m \times 1)$ vector of functions given by $f_i(x) = \sum_{j=1}^{k} g_{ij}(x) \cdot r_j(x)$. Then

$$\frac{\partial}{\partial x'} f(x) = \frac{\partial}{\partial x'} G(x) \cdot [I_n \otimes r(x)] + G(x) \frac{\partial}{\partial x'} r(x),$$

where $\otimes$ denotes the Kronecker product and I_n is the identity matrix of dimension n.

Appendix 2

In this appendix we derive the approximative optimal decision functions for output, investment, and consumption for the case of the Cobb-Douglas production function. We have

$$Y_t = K_t^{1-\alpha}(A_t N_t)^\alpha, \tag{A.1a}$$

$$I_t = K_{t+1} - (1-\delta)K_t, \tag{A.1b}$$

$$C_t = Y_t - I_t. \tag{A.1c}$$

Let $\tilde{Y}_t = \log\left[Y_t/(1+\mu)^t\right]$, $\tilde{C}_t = \log\left[C_t/(1+\mu)^t\right]$, and $\tilde{I}_t = \log\left[I_t/(1+\mu)^t\right]$ be the logarithmic detrended variables. Equations (A.1) can be transformed to

$$\tilde{Y}_t = (1-\alpha)\tilde{K}_t + \alpha\left(\tilde{A}_t + \tilde{N}_t\right), \tag{A.2a}$$

$$\exp\{\tilde{I}_t\} = (1+\mu)\exp\{\tilde{K}_{t+1}\} - (1-\delta)\exp\{\tilde{K}_t\}, \tag{A.2b}$$

$$\exp\{\tilde{C}_t\} = \exp\{\tilde{Y}_t\} - \exp\{\tilde{I}_t\}. \tag{A.2c}$$

The deterministic equilibrium values of these variables, $\tilde{Y}^*$, $\tilde{C}^*$, and $\tilde{I}^*$, are given by

$$\tilde{Y}^* = (1-\alpha)\tilde{K}^* + \alpha\tilde{N}^*, \tag{A.3a}$$

$$\exp\{\tilde{I}^*\} = (\mu+\delta)\exp\{\tilde{K}^*\}, \tag{A.3b}$$

$$\exp\{\tilde{C}^*\} = \exp\{\tilde{Y}^*\} - \exp\{\tilde{I}^*\}. \tag{A.3c}$$

Define $y_t = \tilde{Y}_t - \tilde{Y}^*$, $i_t = \tilde{I}_t - \tilde{I}^*$, $c_t = \tilde{C}_t - \tilde{C}^*$, $K^* = \exp\{\tilde{K}^*\}$, $Y^* = \exp\{\tilde{Y}^*\}$, $I^* = \exp\{\tilde{I}^*\}$, and $C^* = \exp\{\tilde{C}^*\}$. The first order Taylor approximations of equations (A.2) around the values in (A.3) are

$$y_t = (1-\alpha)k_t + \alpha(a_t + n_t), \tag{A.4a}$$

$$I^* i_t = (1+\mu)K^* k_{t+1} - (1-\delta)K^* k_t, \tag{A.4b}$$

$$C^* c_t = Y^* y_t - I^* i_t. \tag{A.4c}$$

Assume that a_t follows a first order autoregressive process. (The case of a higer order process is straightforward.) The optimal decision functions for capital and labor can be written as

$$k_{t+1} = \eta_{kk} k_t + \eta_{ka} a_t, \tag{A.5a}$$

$$n_t = \eta_{nk} k_t + \eta_{na} a_t. \tag{A.5b}$$

Then we have from equations (A.4) and (A.5) the following optimal linear decision rules for output, investment, and consumption:

$$y_t = (1-\alpha+\alpha\eta_{nk})k_t + \alpha(1+\eta_{na})a_t = \eta_{yk}k_t + \eta_{ya}a_t, \tag{A.6a}$$

$$i_t = [(1+\mu)\eta_{kk} - (1-\delta)]\frac{K^*}{I^*}k_t + (1+\mu)\frac{K^*}{I^*}\eta_{ka}a_t = \eta_{ik}k_t + \eta_{ia}a_t, \tag{A.6b}$$

$$c_t = \left(\frac{Y^*}{C^*}\eta_{yk} - \frac{I^*}{C^*}\eta_{ik}\right)k_t + \left(\frac{Y^*}{C^*}\eta_{ya} - \frac{I^*}{C^*}\eta_{ia}\right)a_t = \eta_{ck}k_t + \eta_{ca}a_t. \tag{A.6c}$$

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